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On the $\bar{\partial}$-Neumann Problem for Generalized Functions

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Using the new theory of generalized functions developed by one of the authors the $\bar{\partial}$ equation in $\mathbb{C}^n$ is studied. In particular it is proven that if G is any generalized function on $\mathbb{C}$ (in the above sense) then there is a generalized function S on $\mathbb{C}$ such that $\partial S/\partial \bar{z} = G$. Several other results are proven valid in polydiscs of $\mathbb{C}^n$, for which differential forms whose coefficients are generalized functions are introduced.

INTRODUCTION

In order to solve nonlinear problems (general multiplication of distributions and physical applications) one of the authors (J.F.C.) introduced a new theory of generalized functions more general than distributions; see Colombeau [1 4]. The generalized functions F on $\Omega \subset \mathbb{C}^n$ solution of the equations $\partial F/\partial \bar{z}_i = 0$ $(i = 1,..., n)$ are studied in Colombeau and Galé [5] and called the "holomorphic generalized functions." We study the $\bar{\partial}$-Neumann problem in the setting of generalized functions and we obtain generalizations of many basic classical results, which show the richness of the new theory. The main proofs are more difficult than the classical ones.

In the first part of this paper we introduce a simplification in the definition of the concepts of generalized functions as defined in Colombeau

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[1–3], which makes their use easier. A detailed account of this simplified concepts is given in Sections 1 to 5.

In the following sections we solve the $\bar{\partial}$ problem in various circumstances. The one dimensional case is first solved in the whole space when the second member has compact support, then without the compact support assumption. This latter proof is much more technical than the corresponding classical proof. In the case of several complex variables we introduce "generalized differential forms," i.e., differential forms whose coefficients are generalized functions (therefore they generalize the currents). Then we solve (in the n-dimensional case) the $\bar{\partial}$ problem with compact support, without compact support and we obtain a Dolbeault Grothendieck lemma.

We use the classical notations of Hörmander [6] and Schwartz [7] and for the new generalized functions we use the notations of Colombeau [1–4]. We recall some of them.

1. The Algebras $\mathscr{E}[X]$ and $\mathscr{E}[\Omega]$

Ω denotes an open subset of $\mathbb{R}^n$, and if $\varphi \in \mathscr{D}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\varepsilon > 0$ we recall that we set

$$\varphi_{\varepsilon,x}(\lambda) = \frac{1}{\varepsilon^n}\varphi\left(\frac{\lambda - x}{\varepsilon}\right)$$

for all $\lambda \in \mathbb{R}^n$. If $x = 0$ we set φ_ε in place of $\varphi_{\varepsilon,0}$. We recall that for $q = 1, 2, \ldots$ we set

$$\mathscr{A}_q = \left\{ \varphi \in \mathscr{D}(\mathbb{R}^n) \text{ such that } \int \varphi(x)\,dx = 1 \text{ and } \int x^i \varphi(x)\,dx = 0 \right.$$
$$\left. \text{if } i = (i_1, \ldots, i_n) \in \mathbb{N}^n \text{ with } 1 \leqslant |i| \leqslant q \right\}.$$

We know (Colombeau [1, 3]) or it is immediate that $\mathscr{A}_q \neq \varnothing\ \forall q$,

$$\mathscr{A}_{q+1} \subset \mathscr{A}_q\ \forall q, \bigcap_{q \in \mathbb{N}^*} \mathscr{A}_q = \varnothing, \varphi \in \mathscr{A}_q \Rightarrow \varphi_\varepsilon \in \mathscr{A}_q\ \forall \varepsilon > 0.$$

We denote by $\mathscr{X}$ the set of all subsets X of $\mathscr{D}(\mathbb{R}^n) \times \Omega$ which have the following property:

$$\begin{aligned} &\forall K \subset\subset \Omega\ \exists N \in \mathbb{N} \text{ such that } \forall \varphi \in \mathscr{A}_N \\ &\exists \eta \in]0, 1] \text{ such that } \{\varphi_\varepsilon \mid 0 < \varepsilon < \eta\} \times K \subset X \end{aligned} \tag{1}$$

(as usual the notation $K \subset \subset \Omega$ means that K is a compact subset of Ω). If $X \in \mathscr{X}$ is given we are going to associate to it a new set $X' \in \mathscr{X}$, $X' \subset X$. In some intuitive sense the set X' might be considered as some "interior" of X. For the definition of X' we need a few more notations. For fixed X if $K \subset \subset \Omega$ we set $N(K)$ as the minimum of the numbers N such that (1) holds. For fixed X, if $K \subset \subset \Omega$ and if $\varphi \in \mathscr{A}_{N(K)}$ we set $\eta(K, \varphi)$ to be the supremum of the numbers η such that (1) holds. Now to $X \in \mathscr{X}$ we associate the set X' defined by:

$$X' = \bigcup_{K \subset \subset \Omega} \bigcup_{\varphi \in \mathscr{A}_{N(K)}} (\{\varphi_\varepsilon \mid 0 < \varepsilon < \eta(K, \varphi)\} \times \mathring{K}) \tag{2}$$

(where $\mathring{K}$ denotes as usual the interior of K). $X' \subset X$ and if $(\psi, x) \in X'$ (where $\psi \in \mathscr{D}(\mathbb{R}^n)$ and $x \in \Omega$) then there is a neighborhood ω of x in Ω such that the set $(\{\psi_\varepsilon \mid 0 < \varepsilon < 1\} \times \omega)$ is still contained in X', which justifies the above intuitive conception of X' as some "interior" of X. The following results are immediate:

(a) Any arbitrary union of elements of $\mathscr{X}$ is still an element of $\mathscr{X}$, $\mathscr{X} \neq \varnothing$ and $\varnothing \notin \mathscr{X}$.

(b) The set $\mathscr{X}$ is directed with respect to the relation $\supset$; any finite intersection of elements of $\mathscr{X}$ is still an element of $\mathscr{X}$.

(c) If $X \in \mathscr{X}$ then $X' \in \mathscr{X}$, $X' \subset X$, and as a consequence the subset $\mathscr{X}'$ of $\mathscr{X}$ defined by

$$\mathscr{X}' = \{X' \mid X \in \mathscr{X}\}$$

is cofinal in $\mathscr{X}$.

If $X \in \mathscr{X}$ and $f \in \mathbb{C}^X$ then for all $K \subset \subset \Omega$ (with $\mathring{K} \neq \varnothing$), for all $\varphi \in \mathscr{A}_{N(K)}$ and all ε with $0 < \varepsilon < \eta(K, \varphi)$ we define a function $f_K(\varphi_\varepsilon, \cdot)$ on $\mathring{K}$ by

$$f_K(\varphi_\varepsilon, x) = f(\varphi_\varepsilon, x) \in \mathbb{C}. \tag{3}$$

DEFINITION 1. If $X \in \mathscr{X}$ (Ω is fixed from the beginning) we set $\mathscr{E}[X] = \{f \in \mathbb{C}^X$ such that $\forall K \subset \subset \Omega$ (with $\mathring{K} \neq \varnothing$), $\forall \varphi \in \mathscr{A}_{N(K)}$ and $\forall \varepsilon$ with $0 < \varepsilon < \eta(K, \varphi)$ then $f_K(\varphi_\varepsilon, \cdot) \in \mathscr{C}^\infty(\mathring{K})\}$.

The following results are immediate to prove. $\mathscr{E}[X]$ is a subalgebra of $\mathbb{C}^X$. If X, $Y \in \mathscr{X}$, $X \supset Y$ and $f \in \mathscr{E}[X]$ then the restriction $f|_Y$ is an element of $\mathscr{E}[Y]$. Therefore via these restriction maps the system $(\mathscr{E}[X], \theta_{YX})$, where $\theta_{YX}: \mathscr{E}[X] \to \mathscr{E}[Y]$ denotes the restriction map, is a direct system of algebras.

DEFINITION 2. We set

$$\mathscr{E}[\Omega]=\varinjlim_{X\in\mathscr{X}}\mathscr{E}[X].$$

We denote by θ_X the canonical map $\mathscr{E}[X]\to\mathscr{E}[\Omega]$. Since $\mathscr{X}'$ is a cofinal set in $\mathscr{X}$ then we have

$$\mathscr{E}[\Omega]=\varinjlim_{Y\in\mathscr{X}'}\mathscr{E}[X].$$

Remark. $\mathscr{E}[\Omega]$ may be interpreted as a space of germs: if $f\in\mathscr{E}[X]$ and $g\in\mathscr{E}[Y]$ (for $X, Y\in\mathscr{X}$) then $\theta_X(f)=\theta_Y(g)$ means that there is some element Z of $\mathscr{X}$ such that $f|_Z=g|_Z$.

2. PARTIAL DERIVATIVES OF ELEMENTS OF $\mathscr{E}[X]$ AND $\mathscr{E}[\Omega]$

$\Omega\subset\mathbb{R}^n$ and $X\in\mathscr{X}$ are fixed; if $i=(i_1,\ldots,i_n)\in\mathbb{N}^n$ we consider the derivation operator

$$D=\partial^i=\frac{\partial^{|i|}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}.$$

For any $f\in\mathscr{E}[X]$ we are going to define an element Df of $\mathscr{E}[X']$, where $X'\in\mathscr{X}'$ is the set associated to X in Section 1. If (ψ,y) is an arbitrary element of X' we have $\psi=\varphi_\varepsilon$ and $y\in\mathring{K}$ with $K\subset\subset\Omega$, $\varphi\in\mathscr{A}_{N(K)}$ and $0<\varepsilon<\eta(K,\varphi)$ (note that since $(\varphi_{\varepsilon_1})_{\varepsilon_2}=\varphi_{\varepsilon_1\varepsilon_2}$ the choice of φ and ε is not unique but it does not matter). Since $f\in\mathscr{E}[X]$ the function $f_K(\varphi_\varepsilon,\cdot)$ is in $\mathscr{C}^\infty(\mathring{K})$ and therefore we may define

$$(Df)(\psi,y)=(Df_K(\varphi_\varepsilon,\cdot))(y) \tag{4}$$

which defines a function $Df\in\mathbb{C}^{X'}$. It is clear that $Df\in\mathscr{E}[X']$. In this way we define a linear map D from $\mathscr{E}[X]$ into $\mathscr{E}[X']$ and clearly the usual (Leibnitz) formula for the derivation of a product holds.

Now we are going to define partial derivatives of elements of $\mathscr{E}[\Omega]$; Ω remains fixed but now X ranges in $\mathscr{X}$. It is obvious that if $X, Y\in\mathscr{X}$, $f\in\mathscr{E}[X]$, $g\in\mathscr{E}[Y]$, if, furthermore, $\theta_X(f)=\theta_Y(g)$ then $\theta_{X'}(Df)=\theta_{Y'}(Dg)$. In this way we define $D\Phi$ for any element Φ of $\mathscr{E}[\Omega]$. D is a linear map from $\mathscr{E}[\Omega]$ into $\mathscr{E}[\Omega]$ and Leibnitz's rule for the derivation of a product holds.

3. The Algebra $\mathcal{G}^*(\Omega)$

In this section we define an algebra of "generalized functions" on Ω that we denote by $\mathcal{G}^*(\Omega)$.

Definition 3. An element Φ of $\mathcal{E}[\Omega]$ is said to be *moderate* if there are $X \in \mathcal{X}$ and $f \in \mathcal{E}[X]$ with $\theta_X(f) = \Phi$ such that the following property holds

> $\forall K \subset\subset \Omega$ and $\forall D$ (derivation operator as above) $\exists N \in \mathbb{N}$ such that $\forall \varphi \in \mathcal{A}_N$ $\exists \eta \in]0, 1]$ and $c > 0$ such that
>
> $$|(Df)(\varphi_\varepsilon, x)| \leqslant c\varepsilon^{-N} \tag{5}$$
>
> $\forall x \in K$ and $0 < \varepsilon < \eta$.

Obviously we choose N and η in (5) such that $(\varphi_\varepsilon, x) \in X'$. It is immediate to check that if $Y \in \mathcal{X}$ and $g \in \mathcal{E}[Y]$ is such that $\theta_Y(g) = \Phi$ then (5) still holds for g. We set

$$\mathcal{E}_M[\Omega] = \{\Phi \in \mathcal{E}[\Omega] \text{ which are moderate}\}.$$

Clearly $\mathcal{E}_M[\Omega]$ is a subalgebra of $\mathcal{E}[\Omega]$ and $D(\mathcal{E}_M[\Omega]) \subset \mathcal{E}_M[\Omega]$.

Definition 4. An element Φ of $\mathcal{E}[\Omega]$ is said to be *null* if there are $X \in \mathcal{X}$ and $f \in \mathcal{E}[X]$ with $\theta_X(f) = \Phi$ such that the following property holds

> $\forall K \subset\subset \Omega$ and $\forall D$(derivation) $\exists N \in \mathbb{N}$ such that if $q \geqslant N$ and $\varphi \in \mathcal{A}_q$ $\exists \eta \in]0, 1]$ and $c > 0$ such that (6)
>
> $$|(Df)(\varphi_\varepsilon, x)| \leqslant c\varepsilon^{q-N}$$
>
> $\forall x \in K$ and $0 < \varepsilon < \eta$.

Obviously we choose N and η in (6) such that $(\varphi_\varepsilon, x) \in X'$. It is immediate to check that if $Y \in \mathcal{X}$ and $g \in \mathcal{E}[Y]$ is such that $\theta_Y(g) = \Phi$ then (6) holds for g. We denote by $\mathcal{N}_\Omega$ the set of all $\Phi \in \mathcal{E}[\Omega]$ which are null. Clearly $\mathcal{N}_\Omega$ is an ideal of $\mathcal{E}_M[\Omega]$ and we have $D\mathcal{N}_\Omega \subset \mathcal{N}_\Omega$.

Definition 5. We denote by $\mathcal{G}^*(\Omega)$ the quotient algebra

$$\mathcal{G}^*(\Omega) = \frac{\mathcal{E}_M[\Omega]}{\mathcal{N}_\Omega}.$$

An element of $\mathcal{G}^*(\Omega)$ may be written as $\Phi + \mathcal{N}_\Omega$ for some $\Phi \in \mathcal{E}_M[\Omega]$. If D is any derivation as above then $D(\Phi + \mathcal{N}_\Omega) \subset D\Phi + D\mathcal{N}_\Omega$ therefore the operator D is canonically defined as a linear map from $\mathcal{G}^*(\Omega)$ into $\mathcal{G}^*(\Omega)$. Leibnitz's rule for the derivation of a product holds since this was also true in $\mathcal{E}[\Omega]$.

Connection with the Algebra $\mathscr{G}(\Omega)$ Defined in Colombeau [3]

An algebra $\mathscr{G}(\Omega)$ was defined in Colombeau [3] as the quotient $\mathscr{E}_M(\Omega_{\mathscr{S}(\Omega)})/\mathscr{N}$. With notations of this last paper the following relations are obvious (the first inclusion being the map $[(\varphi_{\varepsilon,x}) \to \Phi(\varphi_{\varepsilon,x})] \to [(\varphi_\varepsilon, x) \to \Phi(\varphi_\varepsilon, x)]$ written on representatives of the elements of the inductive limits)

$$\mathscr{E}(\Omega_{\mathscr{S}(\Omega)}) \subset \mathscr{E}[\Omega]$$

$$\mathscr{E}_M(\Omega_{\mathscr{S}(\Omega)}) = \mathscr{E}_M[\Omega] \cap \mathscr{E}(\Omega_{\mathscr{S}(\Omega)}) \subset \mathscr{E}_M[\Omega]$$

$$\mathscr{N}(\text{of Colombeau [3]}) = \mathscr{N}_\Omega \cap \mathscr{E}(\Omega_{\mathscr{S}(\Omega)}) \subset \mathscr{N}_\Omega$$

$$\mathscr{G}(\Omega) \subset \mathscr{G}^*(\Omega).$$

It appears that the definition of $\mathscr{G}^*(\Omega)$ is some natural evolution and generalization of that of $\mathscr{G}(\Omega)$ by dropping some unessential restriction in the definition of $\mathscr{G}(\Omega)$. However, in order to understand the situation one had to study previously the properties of the elements of $\mathscr{G}(\Omega)$; see Colombeau [1-4]. The elements of $\mathscr{G}^*(\Omega)$ are much more convenient than the ones of $\mathscr{G}(\Omega)$, both for the mathematical study of their theory and for their applications. See the book by Colombeau [4] for more details on these facts. In this paper we shall only be concerned with the study of the $\bar\partial$ equation—that we shall do in the setting of $\mathscr{G}^*(\mathbb{C}^n)$ and for this we need to know a bit more on the elements of $\mathscr{G}^*(\Omega)$, which is the purpose of the following two sections.

Remark. As a consequence $\mathscr{D}'(\Omega)$ is contained in $\mathscr{G}^*(\Omega)$ in the following way: if $T \in \mathscr{D}'(\Omega)$, $\varphi \in \mathscr{A}_q$, $0 < \varepsilon < 1$ and $x \in \Omega$ we set $f(\varphi_\varepsilon, x) = \langle T, \varphi_{\varepsilon,x} \rangle$ as soon as $\varphi_{\varepsilon,x} \in \mathscr{D}(\Omega)$. This last condition amounts to $(\varphi_\varepsilon, x) \in X$ for some suitable set $X \in \mathscr{X}$ (depending on the diameter of the support of φ, of $\varepsilon > 0$ and of the distance from x to the boundary of Ω). It is proved in Colombeau [1, 3] that f is moderate and that if f is in $\mathscr{N}(\Omega)$ then $T = 0$ in $\mathscr{D}'(\Omega)$. The distribution T is identified with the class of f in $\mathscr{G}^*(\Omega)$.

4. Support of Elements of $\mathscr{G}^*(\Omega)$

Let U and V denote two open subsets of $\mathbb{R}^n$ such that $U \subset V$; then we have a canonical map $X \in \mathscr{X}_V \mapsto X \cap [\mathscr{D}(\mathbb{R}^n) \times U] \in \mathscr{X}_U$ if $\mathscr{X}_V$ and $\mathscr{X}_U$ denote the sets $\mathscr{X}$ respectively associated to V and U. We have a canonical morphism

$$j_U^V : \mathscr{E}[V] \longrightarrow \mathscr{E}[U].$$

Clearly we have $j_U^V(\mathcal{N}_V) \subset \mathcal{N}_U$ and $j_U^V(\mathcal{E}_M[V]) \subset \mathcal{E}_M[U]$. Therefore we have a canonical morphism

$$J_U^V : \mathcal{G}^*(V) \longrightarrow \mathcal{G}^*(U)$$

which is clearly consistent with the derivation. If $F \in \mathcal{G}^*(V)$ then $J_U^V(F) \in \mathcal{G}^*(U)$ is naturally called the restriction of F to U, and we denote it by $F|_U$. If $F|_U = 0$ we say that F is null on U. If V_1 and V_2 are two open subsets of $\mathbb{R}^n$, if $F \in \mathcal{G}^*(V_1)$ and $G \in \mathcal{G}^*(V_2)$, we say that F and G coincide on $U \subset V_1 \cap V_2$ if $F|_U = G|_U$ in $\mathcal{G}^*(U)$. Clearly if F, $G \in \mathcal{G}^*(U)$, if $U = \bigcup_{\lambda \in \Lambda} U_\lambda$, where U_λ are non-void open sets and if $F|_{U_\lambda} = G|_{U_\lambda}$ for all index $\lambda \in \Lambda$ then $F = G$ in $\mathcal{G}^*(U)$(i.e., $\mathcal{G}^*$ is a presheaf which satisfies the first axom of sheaves); the proof is immediate from the definitions.

As a consequence if $F \in \mathcal{G}^*(\Omega)$ the union of all open sets $U \subset \Omega$ such that $F|_U = 0$ is an open set in which F is null, and it is obviously the largest open set in which F is null. Its complement is called, as usual for distributions, the support of F and is denoted by supp F.

5. Extension of Domains

Let us consider the following subset of $\mathcal{A}_q$

$$\mathcal{A}'_q = \{\varphi \in \mathcal{A}_q \text{ such that } \operatorname{diam}(\operatorname{supp} \varphi) = 1\}.$$

Since $(\varphi_{\varepsilon_1})_{\varepsilon_2} = \varphi_{\varepsilon_1 \varepsilon_2}$ it is immediate to check that for any $\psi \in \mathcal{A}_q$ there exist unique $\varphi \in \mathcal{A}_q$ and $\varepsilon > 0$ such that $\psi = \varphi_\varepsilon$. In this section Ω denotes as usual a fixed open subset of $\mathbb{R}^n$. The aim of this section is to prove the following result, which states that any $G \in \mathcal{G}^*(\Omega)$ has a representative defined on $\mathcal{A}_1 \times \Omega$. Any $G \in \mathcal{G}^*(\Omega)$ is defined as a class of elements of $\mathcal{E}[\Omega] = \varinjlim_{X \in \mathcal{X}} \mathcal{E}[X]$; the following result states that we may replace $\mathcal{E}[\Omega]$ by $\mathcal{E}[\mathcal{A}_1 \times \Omega]$ ($\mathcal{A}_1 \times \Omega$ is the largest useful set $X \in \mathcal{X}$) which will bring important simplifications in the exposition of the theory. However, we believe it is necessary to give the definitions in Sections 1 and 2 using the sets $\mathcal{X}$ and an inductive limit in order to make clear the connection with the concept exposed in Colombeau [3] and the inclusion $\mathcal{G}(\Omega) \subset \mathcal{G}^*(\Omega)$.

THEOREM 1. *For every $G \in \mathcal{G}^*(\Omega)$ there exists a representative $g \in \mathcal{E}[\mathcal{A}_1 \times \Omega]$ (i.e., g is a map from $\mathcal{A}_1 \times \Omega$ to $\mathbb{C}$ and for every $\psi \in \mathcal{A}_1$ the function $x \to g(\psi, x)$ is C^∞ in Ω).*

Proof. We denote by $(K_n)_{n \geqslant 1}$ an exhaustive sequence of compact subsets of Ω with $K_n \subset \mathring{K}_{n+1}$ for all n, and $\mathring{K}_1 \neq \emptyset$. We denote by $(\alpha_n)_{n \geqslant 1}$ a sequence of elements of $\mathscr{D}(\Omega)$ such that $0 \leqslant \alpha_n(x) \leqslant 1$ for all x, $\alpha_n \equiv 1$ on K_{n-1} and $\operatorname{supp} \alpha_n \subset \mathring{K}_n$ for all $n \geqslant 2$.

Let $X \in \mathscr{X}$ and $f \in \mathscr{E}[X]$ be a representative of G. By definition and since $\mathring{K}_\nu \neq \varnothing$ for all ν, there is $N_\nu \geqslant 1$ such that for all $\varphi \in \mathscr{A}'_{N_\nu}$ there is $\eta(\nu, \varphi) > 0$ such that if $0 < \varepsilon < \eta(\nu, \varphi)$ then $f_{\mathring{K}_\nu}(\varphi_\varepsilon, \cdot) \in \mathscr{C}^\infty(\mathring{K}_\nu)$. Therefore the following assertions $(A_\nu)_{\nu=1,2,\ldots}$ hold:

(A_1) $\exists N_1 \geqslant 1$ such that $\forall \varphi \in \mathscr{A}'_{N_1}$ $\exists \eta(1, \varphi) > 0$ such that $f_{\mathring{K}_1}(\varphi_\varepsilon, \cdot) \in \mathscr{C}^\infty(\mathring{K}_1)$ if $0 < \varepsilon < \eta(1, \varphi)$.

$\vdots$

(A_ν) $\exists N_\nu > N_{\nu-1}$ such that $\forall \varphi \in \mathscr{A}'_{N_\nu}$ $\exists \eta(\nu, \varphi) > 0$ such that $f_{\mathring{K}_\nu}(\varphi_\varepsilon, \cdot) \in \mathscr{C}^\infty(\mathring{K}_\nu)$ if $0 < \varepsilon < \eta(\nu, \varphi)$.

Now we are going to define from f the map $g: \mathscr{A}_1 \times \Omega \to \mathbb{C}$. If $(\psi, x) \in \mathscr{A}_1 \times \Omega$ we know there are unique $\varphi \in \mathscr{A}'_1$ and $\varepsilon > 0$ such that $\psi = \varphi_\varepsilon$. We are going to consider several possible cases.

First case. $\varphi \notin \mathscr{A}'_{N_1}$. Then we set $g(\psi, y) = g(\varphi_\varepsilon, y) = 0$ for all $y \in \Omega$.

Second case. $\varphi \in \mathscr{A}'_{N_1}$. Since $\mathscr{A}'_{N_{\nu+1}} \subsetneqq \mathscr{A}'_{N_\nu}$ and $\bigcap_{\nu \in \mathbb{N}^*} \mathscr{A}'_{N_\nu} = \varnothing$ there is a maximum number $\nu = m(\varphi)$ such that $\varphi \in \mathscr{A}'_{N_{m(\varphi)}}$ and $\varphi \notin \mathscr{A}'_{N_{m(\varphi)+1}}$. We consider the real number $\eta(m(\varphi), \varphi)$ in assertion $(A_{m(\varphi)})$ above. We know that $f_{\mathring{K}_{m(\varphi)}}(\varphi_\delta, \cdot) \in \mathscr{C}^\infty(\mathring{K}_{m(\varphi)})$ if $0 < \delta < \eta(m(\varphi), \varphi)$. We have the two possibilities (a) and (b) below.

Possibility (a). $\varepsilon \geqslant \eta(m(\varphi), \varphi)$. Then we set $g(\psi, y) = g(\varphi_\varepsilon, y) = 0$ for all $y \in \Omega$.

Possibility (b). $0 < \varepsilon < \eta(m(\varphi), \varphi)$. Then we set

$$g(\psi, y) = g(\varphi_\varepsilon, y) = (\alpha_{m(\varphi)}(y)) \cdot f(\varphi_\varepsilon, y)$$

for all $y \in \Omega$. The mapping $(y \to f(\varphi_\varepsilon, y))$ is $\mathscr{C}^\infty$ in $\mathring{K}_{m(\varphi)}$ and supp. $\alpha_{m(\varphi)} \subset \mathring{K}_{m(\varphi)}$ therefore $g(\psi, \cdot) \in \mathscr{C}^\infty(\Omega)$. Since $\alpha_{m(\varphi)} \equiv 1$ in $K_{m(\varphi)-1}$ we have $g(\varphi_\varepsilon, y) = f(\varphi_\varepsilon, y)$ for all $y \in K_{m(\varphi)-1}$.

Therefore our function g defined in the above cases has domain $\mathscr{A}_1 \times \Omega$. Furthermore for each fixed $\psi \in \mathscr{A}_1$ the function $g(\psi, \cdot)$ is in $\mathscr{C}^\infty(\Omega)$. Therefore $g \in \mathscr{E}[\mathscr{A}_1 \times \Omega]$. From the last remark in Possibility (b) it follows that g coïncides with f in some set $Y \in \mathscr{X}$, $Y \subset X$. ∎

Therefore the inductive limit may be dropped in the construction of $\mathscr{G}^*(\Omega)$, and we may replace everywhere in it $\mathscr{E}[\Omega]$ by $\mathscr{E}[\mathscr{A}_1 \times \Omega]$. We define $\mathscr{E}_M[\mathscr{A}_1 \times \Omega]$ and $\mathscr{N}[\mathscr{A}_1 \times \Omega]$ from $\mathscr{E}[\mathscr{A}_1 \times \Omega]$ by properties (5) and (6), respectively (see Definitions 3 and 4), and we have $\mathscr{G}^*(\Omega) = \mathscr{E}_M[\mathscr{A}_1 \times \Omega]/\mathscr{N}[\mathscr{A}_1 \times \Omega]$.

6. The Equation $\partial S/\partial \bar{z} = G$, $G \in \mathscr{G}^*(\mathbb{C})$ with Compact Support

THEOREM 2. *For every $G \in \mathscr{G}^*(\mathbb{C})$ with compact support there is an $S \in \mathscr{G}^*(\mathbb{C})$ such that $\partial S/\partial \bar{z} = G$.*

The proof will follow from the following lemmas.

LEMMA 1. *If $\varphi \in \mathscr{D}(\mathbb{R}^2)$ and $K = \operatorname{supp} \varphi$ then the integral*

$$u(\zeta) = \frac{1}{2\pi i} \int \frac{\varphi(z)}{z - \zeta} \, dz \wedge d\bar{z}$$

defines a function u which is analytic in $\mathbb{C} - K$, C^∞ in $\mathbb{C}$ and such that $\partial u/\partial \bar{z} = \varphi$ in $\mathbb{C}$.

It is a particular case of Theorem 1.2.2. in Hörmander [6].

The following result shows that every generalized function with compact support admits a representative with a "good" support property.

LEMMA 2. *Let Ω be an open subset of $\mathbb{R}^n$. Then for every $G \in \mathscr{G}^*(\Omega)$ with compact support K and for every compact set $L \subset\subset \Omega$ with $K \subset \mathring{L}$ there is a representative $g \in \mathscr{E}_M[\mathscr{A}_1 \times \Omega]$ of G such that for all $\psi \in \mathscr{A}_1$ the C^∞ function $g(\psi, \cdot)$ is in $\mathscr{D}(\mathring{L})$.*

Proof. From Theorem 1 there is a representative $f \in \mathscr{E}_M[\mathscr{A}_1 \times \Omega]$ of G. Let $\alpha \in \mathscr{D}(\Omega)$ be such that $0 \leqslant \alpha \leqslant 1$, $\alpha \equiv 1$ in an open neighborhood V of K and $\operatorname{supp} \alpha \subset \mathring{L}$. If $\beta = 1 - \alpha$ then α, $\beta \in \mathscr{E}(\Omega) \subset \mathscr{G}^*(\Omega)$. Therefore in $\mathscr{G}^*(\Omega)$ we have

$$G = 1 \cdot G = (\alpha + \beta)\, G = \alpha G + \beta G.$$

Let us prove that $\beta G = 0$ in $\mathscr{G}^*(\Omega)$; i.e., that $\beta f \in \mathscr{N}[\mathscr{A}_1 \times \Omega]$. For this we have to prove that:

> $\forall H \subset\subset \Omega$ and $\forall D$ (derivation) $\exists N \in \mathbb{N}$ such that if $\varphi \in \mathscr{A}_q$ with $q \geqslant N$ $\exists \eta \in]0, 1]$ and $c > 0$ such that $|D(\beta f)(\varphi_\varepsilon, x)| \leqslant c(\varepsilon)^{q-N}$ if $x \in H$ and $0 < \varepsilon < \eta$. (P)

We shall prove (P) by noticing that $H = (H \cap V) \cup (H \cap \complement V)$ and by proving the above inequality in $(H \cap V)$ and in $(H \cap \complement V)$. It is proved in Colombeau–Galé [5] that the function $(\varphi_\varepsilon, x) \to \beta(x)$ is a representative of β in $\mathscr{G}(\Omega)$ therefore in $\mathscr{G}^*(\Omega)$. Since $\beta = 0$ in V (as an element of $\mathscr{E}(V)$) then $(\beta f)(\varphi_\varepsilon, x) = 0$ if $x \in V$ therefore $(D(\beta f))(\varphi_\varepsilon, x) = 0$ if $x \in V$ which gives a fortiori the desired inequality in $H \cap V$. Now $G = 0$ in $\complement K$ which is an open neighborhood of $H \cap \complement V$. $H \cap \complement V$ is a compact subset of the open set $\complement K$ (in

Ω) and since $(\beta G)|_{\complement K} = 0$ in $\mathscr{G}^*(\complement K)$ we have the desired inequality in $H \cap \complement V$. Therefore (P) holds. Therefore $\beta G = 0$ in $\mathscr{G}^*(\Omega)$, i.e., $G = \alpha G$ in $\mathscr{G}^*(\Omega)$ and therefore $g = \alpha f$ (i.e., $g(\varphi_\varepsilon, x) = \alpha(x) f(\varphi_\varepsilon, x)$) is a representative of G. ∎

LEMMA 3. *Let be given $G \in \mathscr{G}^*(\Omega)$ with compact support K, let L be a compact subset of $\mathbb{C}$ such that $K \subset \mathring{L}$ and let $g \in \mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}]$ be a representative of G having the property of Lemma 2. Then the function $g^*: \mathscr{A}_1 \times \mathbb{C} \to \mathbb{C}$ defined by*

$$g^*(\psi, \zeta) = \frac{1}{2\pi i} \int \frac{g(\psi, z)}{z - \zeta} \, dz \wedge d\bar{z} \tag{6}$$

(Lemma 1) is in $\mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}]$ and $\partial g^*/\partial \bar{z} = g$.

Proof. The existence of g^* in $\mathscr{E}[\mathscr{A}_1 \times \mathbb{C}]$ and the equality $(\partial g^*/\partial \bar{z})(\psi, \zeta) = g(\psi, \zeta)$ follow from Lemma 1, ψ playing the role of a parameter. We have to prove that g^* is moderate, i.e.,

$\forall H \subset\subset \mathbb{C}$ and $\forall D\, \exists N \in \mathbb{N}$ such that if $\varphi \in \mathscr{A}_N$ $\exists \eta \in]0, 1]$ and $c > 0$ such that $|Dg^*(\psi, \zeta)| \leqslant c(\varepsilon)^{-N}$ if $0 < \varepsilon < \eta$ and $\zeta \in H$. (P′)

The above inequality without D, (i.e., $D = \text{identity}$), comes from formula (6) since $g \in \mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}]$ is null for $z \notin L$ and thus verifies an inequality as above for all $z \in \mathbb{C}$, and since the function $z \to 1/(z - \zeta)$ is locally integrable. Now if D is not the identity one proves easily by a change of variables that

$$Dg^*(\varphi_\varepsilon, \zeta) = \frac{1}{2\pi i} \int \frac{Dg(\varphi_\varepsilon, z + \zeta)}{z} \, dz \wedge d\bar{z}$$

and the above method of majorization holds. ∎

Proof of Theorem 2. Defining S as the class of g^* we have Theorem 2.

7. THE EQUATION $\partial S/\partial \bar{z} = G$ FOR ARBITRARY G IN $\mathscr{G}^*(\mathbb{C})$

THEOREM 3. *For every $G \in \mathscr{G}^*(\mathbb{C})$ there exists an $S \in \mathscr{G}^*(\mathbb{C})$ such that $\partial S/\partial \bar{z} = G$.*

Proof. If $j \in \mathbb{N}^*$ we set $K_j = \{z \in \mathbb{C}$ such that $|z| \leqslant j\}$. Let $\beta_j \in \mathscr{D}(\mathbb{R}^2)$ be such that $0 \leqslant \beta_j \leqslant 1$, $\beta_j \equiv 1$ in $V_j = \{z$ such that $|z| \leqslant j + \tau\}$, for some fixed τ such that $0 < \tau < 1$. We define a sequence $(\alpha_j)_{j \geqslant 1}$ by $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j - \beta_{j-1}$ if $j > 1$. Then, since $\alpha_j \in \mathscr{D}(\mathbb{R}^2)$, $\alpha_j G \in \mathscr{G}^*(\mathbb{C})$ and has compact support. From Theorem 2 there exists $S_j \in \mathscr{G}^*(\mathbb{C})$ such that $\partial S_j/\partial \bar{z} = \alpha_j G$. Let $f \in \mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}]$ be a representative of G (Theorem 1); $g_j = \alpha_j f$

represents $\alpha_j G$ and therefore it follows from Lemma 3 of Section 6 that S_j as constructed in Theorem 2 admits a representative $u_j \in \mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}]$ given by

$$u_j(\psi, z) = \frac{1}{2\pi i} \int \frac{\alpha_j(\zeta) f(\psi, \zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}$$

and we know that

$$\frac{\partial u_j}{\partial \bar{z}}(\psi, z) = \alpha_j(z) f(\psi, z).$$

If $j > 1$, $\alpha_j \equiv 0$ in V_{j-1} therefore the function $(z \to u_j(\psi, z))$ is holomorphic in V_{j-1} for all $\psi \in \mathscr{A}_1$ and therefore it has a power series expansion

$$u_j(\psi, z) = \sum_{n=0}^{+\infty} (a_n^j(\psi)) \cdot (z)^n \qquad \forall z \in V_{j-1}. \tag{7}$$

For any $\psi \in \mathscr{A}_1$ and $j > 1$ let us choose $N(\psi, j) \in \mathbb{N}$ such that if

$$v_j(\psi, z) = \sum_{n=0}^{N(\psi, j)} (a_n^j(\psi)) \cdot (z)^n \tag{8}$$

we have

$$\sup_{z \in K_{j-1}} |(u_j - v_j)(\psi, z)| \leqslant 2^{-j}. \tag{9}$$

We shall choose more precisely $N(\psi, j)$ later in the proof. At present let us define a function $u\colon \mathscr{A}_1 \times \mathbb{C} \to \mathbb{C}$ by

$$u(\psi, z) = u_1(\psi, z) + \sum_{j \geqslant 2} (u_j - v_j)(\psi, z). \tag{10}$$

Considering ψ as a parameter it follows from Hörmander [6, Theorem 1.4.4] that the funtion $z \to u(\psi, z)$ is in $C^\infty(\mathbb{C})$ for all $\psi \in \mathscr{A}_1$ and that $(\partial u/\partial \bar{z})(\psi, z) = f(z)$ for all $z \in \mathbb{C}$ and all $\psi \in \mathscr{A}_1$. We would like to obtain S in Theorem 3 as the class of u in $\mathscr{G}^*(\mathbb{C})$ and for this it suffices to check that u is moderate; this will follow from a careful choice of the numbers $N(\psi, j)$ considered above. To prove that u is moderate we have to prove that

> $\forall K \subset\subset \mathbb{C}$ and $\forall D$ (derivation) $\exists N \in \mathbb{N}$ such that $\forall \varphi \in \mathscr{A}_N \exists \eta \in]0, 1]$ and $c > 0$ such that $|(Du)(\varphi_\varepsilon, z)| \leqslant c(\varepsilon)^{-N}$ if $z \in K$ and $0 < \varepsilon < \eta$. (11)

In (11) we may assume $K=K_p$ for some $p\in\mathbb{N}^*$. From (10)

$$|(Du)(\psi,z)|\leqslant|(Du_1)(\psi,z)|+\sum_{1<j\leqslant p+1}|D(u_j-v_j)(\psi,z)|$$

$$+\sum_{j>p+1}|D(u_j-v_j)(\psi,z)|. \tag{12}$$

Since u_1 is moderate $(Du_1)(\psi,z)$ satisfies an inequality of type (11). Since $j\geqslant p+2$ it follows from (9) and Cauchy's inequalities that the third term in the second member of (12) is bounded above by a convergent series independent of ψ when $z\in K_p$. In the second term we know from Section 6 that each function u_j is moderate therefore in order to prove (11) it suffices to check that each function v_j is moderate; this will come from a suitable choice of the numbers $N(\psi,j)$ as explained now: for fixed $K=K_p$, D and j we have to prove that

$\exists N\in\mathbb{N}$ such that $\forall\varphi\in\mathscr{A}_N\ \exists\eta\in\,]0,1]$ and $c>0$ such that $|(Dv_j)(\varphi_\varepsilon,z)|\leqslant c(\varepsilon)^{-N}$ if $z\in K_p$ and $0<\varepsilon<\eta$. (13)

First we consider the case when D is the identity map (i.e., the order of D is zero). Since (K_p) is increasing we may assume, for fixed j, that $p>j$; we are going to show that we may choose $N(\psi,j)$ large enough such that (9) holds but at the same time small enough such that (12) holds (in this first case with D equals to identity-- the general case of an arbitrary D will follow; see below the last part of the proof). Since u_j is moderate (and since $\bar{V}_{j-1}$ is compact) there is an $N\in\mathbb{N}$ such that $\forall\varphi\in\mathscr{A}_N\ \exists\eta\in\,]0,1]$ and $c>0$ such that $|u_j(\varphi_\varepsilon,z)|\leqslant c(\varepsilon)^{-N}$ if $0<\varepsilon<\eta$ and $|z|\leqslant j+\tau-1$ (i.e., $z\in\bar{V}_{j-1}$). Since the function $z\to u_j(\varphi_\varepsilon,z)$ is holomorphic in V_{j-1} then it follows from Cauchy's inequalities that

$$|a_n^j(\varphi_\varepsilon)|\leqslant\frac{c}{\varepsilon^N}\frac{1}{(j+\tau-1)^n} \tag{14}$$

if $0<\varepsilon<\eta$ and $n\in\mathbb{N}$. Therefore if $z\in K_{j-1}$

$$\begin{aligned}|(u_j-v_j)(\varphi_\varepsilon,z)|&=\sum_{n>N(\varphi_\varepsilon,j)}|a_n^j(\varphi_\varepsilon)|\,|z|^n\\&\leqslant\sum_{n>N(\varphi_\varepsilon,j)}\frac{c}{\varepsilon^N}\frac{(j-1)^n}{(j+\tau-1)^n}\\&=\sum_{n>N(\varphi_\varepsilon,j)}\frac{c}{\varepsilon^N}\left(\frac{l}{l+\tau}\right)^n\\&=\frac{c}{\varepsilon^N}\left(\frac{l}{l+\tau}\right)^{N(\varphi_\varepsilon,j)+1}\frac{l+\tau}{\tau}=\frac{c}{\varepsilon^N}\frac{l}{\tau}\left(\frac{l}{l+\tau}\right)^{N(\varphi_\varepsilon,j)}\end{aligned}$$

if we set $l=j-1$. This last quantity is less than 2^{-j} (in order to have (9)) as soon as $(l/(l+\tau))^{N(\varphi_\varepsilon,j)} \leqslant 2^{-j}(\varepsilon^N \tau/cl)$.

Setting $c'=\tau/cl$, $k_1=j\operatorname{Log}2-\operatorname{Log}c'$ and $k_2=\operatorname{Log}(l+\tau)-\operatorname{Log}l$ this last inequality amounts to

$$N(\varphi_\varepsilon,j) \geqslant \frac{k_1+N\log(1/\varepsilon)}{k_2}. \tag{15}$$

For $\varepsilon>0$ small enough we have

$$2N\operatorname{Log}\frac{1}{\varepsilon} \geqslant \frac{k_1+N\operatorname{Log}(1/\varepsilon)}{k_2}$$

and therefore (15) holds if we choose

$$N(\varphi_\varepsilon,j)=\left[\!\!\left[2N\operatorname{Log}\frac{1}{\varepsilon}\right]\!\!\right]+1 \tag{16}$$

where we denote by $[\![\alpha]\!]$ the entire part of α, $\alpha\in\mathbb{R}^+$ and with this choice of $N(\varphi_\varepsilon,j)$ we have (9). We are going to check (13) (with $D=$ identity). If $z\in K_p$, i.e., $|z|\leqslant p$, and $0<\varepsilon<\eta$, it follows from (14) and (8) that $(l=j-1)$:

$$\begin{aligned}
|v_j(\varphi_\varepsilon,z)| &\leqslant \sum_{n=0}^{n=N(\varphi_\varepsilon,j)} |a_n^j(\varphi_\varepsilon)|\,|z|^n \\
&\leqslant \sum_{n=0}^{N(\varphi_\varepsilon,j)} \frac{c}{\varepsilon^N}\frac{(p)^n}{(j+\tau-1)^n} \\
&= \frac{c}{\varepsilon^N}\frac{l+\tau}{p-l-\tau}\left[\left(\frac{p}{l+\tau}\right)^{N(\varphi_\varepsilon,j)+1}-1\right] \\
&\leqslant \frac{c'}{\varepsilon^N}\left(\frac{p}{l+\varepsilon}\right)^{N(\varphi_\varepsilon,j)+1}
\end{aligned}$$

(obvious value of c'). If $\delta=(p/(l+\tau))^2$, then $\delta>1$ and the above inequality may be written as

$$|v_j(\varphi_\varepsilon,z)| \leqslant \frac{c'}{\varepsilon^N}(\delta^{1/2})^{N(\varphi_\varepsilon,j)+1}$$

which from (16) gives

$$|v_j(\varphi_\varepsilon,z)| \leqslant \frac{c'}{\varepsilon^N}(\delta)^{N\operatorname{Log}(1/\varepsilon)+1} = \frac{c''}{\varepsilon^N}(\delta)^{N\operatorname{Log}(1/\varepsilon)} \leqslant \frac{c''}{(\varepsilon)^{N+N''}}$$

where $N'' \in \mathbb{N}$ and $N'' \geq N \operatorname{Log} \delta$. Therefore we obtain (13) in the particular case when the order of D is zero. Now we prove (13) when the order of D is > 0. We shall just check that (13) holds with the above choice of $N(\varphi_\varepsilon, j)$. Setting $f_n(z) = z^n$ we have from (8)

$$|(Dv_j)(\psi, z)| \leqslant \sum_{n=0}^{N(\psi, j)} |a_n'(\psi)| \, |Df_n(z)|. \tag{17}$$

If $z \in K_p$ and $0 < \rho < \tau$ then $|\zeta - z| = \rho$ implies $|\zeta| \leqslant p + \tau$. From this remark Cauchy's inequalities give if $z \in K_p$ and $D = (\partial^{|k|}/\partial x^{k_1} \partial y^{k_2})(z = x + iy)$:

$$\begin{aligned} |(Df_n)(z)| &\leqslant \frac{|k|!}{\rho^{|k|}} \sup_{|\zeta - z| = \rho} |f_n(\zeta)| \\ &= \frac{|k|!}{\rho^{|k|}} \sup_{|\zeta - z| = \rho} |\zeta|^n \leqslant \frac{|k|!}{\rho^{|k|}} (p + \tau)^n. \end{aligned}$$

Therefore from (17) and (14) we obtain

$$|(Dv_j)(\varphi_\varepsilon, z)| \leqslant \sum_{n=0}^{N(\varphi_\varepsilon, j)} \frac{c}{\varepsilon^N} \frac{1}{(j + \tau - 1)^n} \frac{|k|!}{\rho^{|k|}} (p + \tau)^n$$

and

$$|(Dv_j)(\varphi_\varepsilon, z)| \leqslant \frac{c'}{\varepsilon^N} \left(\frac{p + \tau}{j + \tau - 1} \right)^{N(\varphi_\varepsilon, j)} \tag{18}$$

if

$$c' = \frac{c \, |k|!}{\rho^{|k|}} \frac{p + \tau}{p - j + 1}.$$

Setting

$$T = \frac{p + \tau}{j + \tau - 1} > 1 \qquad (\text{since } p > j)$$

and using (16) we have

$$T^{N(\varphi_\varepsilon, j)} \leqslant T \cdot \left(\frac{1}{\varepsilon} \right)^{N'}$$

if $N' \geqslant 2N \operatorname{Log} T$, whence from (18) we obtain

$$|(Dv_j)(\varphi_\varepsilon, z)| \leqslant \frac{c'T}{(\varepsilon)^{N + N'}}$$

if $z \in K_p$ and $0 < \varepsilon < \eta$ which proves (13). ∎

8. Generalized Differential Forms

In order to study the $\bar{\partial}$ equation in $\mathbb{C}^n$ with $n \geqslant 2$ we need to introduce "generalized differential forms," i.e., differential forms whose coefficients are generalized functions and which therefore generalize the currents (Schwartz [7]). This is done as an exact generalization of the classical case but we prefer to recall the definitions; in particular this gives a meaning to a general multiplication of the currents. Ω denotes an open subset of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. For all $v \in \mathbb{N}^*$ we set

$$H_v = \{L = (l_1, \dots, l_v) \in \mathbb{N}^v \text{ such that } 1 \leqslant l_1 < l_2 < \cdots < l_v \leqslant n\}$$

and

$$dz^I = dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_r} \qquad \text{if } I = (i_1, \dots, i_r) \in H_r$$

$$d\bar{z}^J = d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \cdots \wedge d\bar{z}_{j_s} \qquad \text{if } J = (j_1, \dots, j_s) \in H_s.$$

We set, if $r, s \in \mathbb{N}^*$,

$$B_{r,s} = \{dz^I \wedge d\bar{z}^J \text{ such that } I \in H_r \text{ and } J \in H_s\}.$$

If $r > 0$ and $s = 0$ we set

$$B_{r,0} = \{dz^I \text{ such that } I \in H_r\}$$

(similar definition in case $r = 0$ and $s > 0$). With these notations we may define the generalized differential forms.

Definition 6. If $(r, s) \in \mathbb{N}^2$ with $r + s > 0$ we call "(r, s)-generalized differential form on Ω" any element of the $\mathscr{G}^*(\Omega)$-free module over $B_{r,s}$. This free module will be denoted by $\mathscr{G}^*_{(r,s)}(\Omega)$. We set $\mathscr{G}^*_{(0,0)}(\Omega) = \mathscr{G}^*(\Omega)$.

Therefore any element of $\mathscr{G}^*_{(r,s)}(\Omega)$ has the form

$$\sum_{I \in H_r, J \in H_s} G_{I,J}\, dz^I \wedge d\bar{z}^J$$

with $G_{I,J} \in \mathscr{G}^*(\Omega)$, and conversely any such object is by definition an element of $\mathscr{G}^*_{(r,s)}(\Omega)$.

Definition 7. We define the linear operators $\partial_v = \partial/\partial z_v$ and $\bar{\partial}_v = \partial/\partial \bar{z}_v$ $(1 \leqslant v \leqslant n)$ from $\mathscr{G}^*(\Omega)$ into $\mathscr{G}^*(\Omega)$ by the usual formulas

$$\partial_v G = \frac{\partial G}{\partial z_v} = \frac{1}{2}\left(\frac{\partial G}{\partial x_v} - i\frac{\partial G}{\partial y_v}\right)$$

$$\bar{\partial}_v G = \frac{\partial G}{\partial \bar{z}_v} = \frac{1}{2}\left(\frac{\partial G}{\partial x_v} + i\frac{\partial G}{\partial y_v}\right)$$

if $z_\nu = (x_\nu, y_\nu) \in \mathbb{R}^2$. We define the linear operators

$$\partial \colon \mathscr{G}^*(\Omega) \longrightarrow G^*_{(1,0)}(\Omega)$$

and

$$\bar{\partial} \colon \mathscr{G}^*(\Omega) \longrightarrow \mathscr{G}^*_{(0,1)}(\Omega)$$

by the formulas

$$\partial G = \sum_{\nu=1}^{n} \frac{\partial G}{\partial z_\nu}\, dz_\nu$$

and

$$\bar{\partial} G = \sum_{\nu=1}^{n} \frac{\partial G}{\partial \bar{z}_\nu}\, d\bar{z}_\nu.$$

DEFINITION 8. We define the exterior product

$$\mathscr{G}^*_{(r,s)}(\Omega) \times \mathscr{G}^*_{(p,q)}(\Omega) \to \mathscr{G}^*_{(r+p,s+q)}(\Omega)$$
$$(G, F) \to G \wedge F$$

by the following: if

$$G = \sum_{I \in H_r, J \in H_s} G_{I,J}\, dz^I \wedge d\bar{z}^J \in \mathscr{G}^*_{(r,s)}(\Omega)$$

$$F = \sum_{K \in H_p, L \in H_q} F_{K,L}\, dz^K \wedge d\bar{z}^L \in \mathscr{G}^*_{(p,q)}(\Omega)$$

then we set

$$G \wedge F = \sum_{\substack{I \in H_r, J \in H_s \\ K \in H_p, L \in H_q}} G_{I,J} \cdot F_{K,L}\, dz^I \wedge d\bar{z}^J \wedge dz^K \wedge d\bar{z}^L$$

(if $r+s>n$ or $p+q>n$ then $G \wedge F = 0$).

Remark. If $G \in \mathscr{G}^*(\Omega)$ and $dz^I \wedge d\bar{z}^J \in B_{r,s} \subset \mathscr{G}^*_{(r,s)}(\Omega)$ then we have

$$\partial G \wedge dz^I \wedge d\bar{z}^J = \left(\sum_{\nu=1}^{n} \frac{\partial G}{\partial z_\nu}\, dz_\nu\right) \wedge dz^I \wedge d\bar{z}^J \in \mathscr{G}^*_{(r+1,s)}(\Omega)$$

and

$$\bar{\partial} G \wedge dz^I \wedge d\bar{z}^J = \left(\sum_{\nu=1}^{n} \frac{\partial G}{\partial \bar{z}_\nu}\, d\bar{z}_\nu\right) \wedge dz^I \wedge d\bar{z}^J \in \mathscr{G}^*_{(r,s+1)}(\Omega).$$

DEFINITION 9. We define the linear operators

$$\partial\colon \mathscr{G}^*_{(r,s)}(\Omega) \to \mathscr{G}^*_{(r+1,s)}(\Omega)$$

and

$$\bar{\partial}\colon \mathscr{G}^*_{(r,s)}(\Omega) \to \mathscr{G}^*_{(r,s+1)}(\Omega)$$

in the following way: if

$$G = \sum_{I \in H_r, J \in H_s} G_{I,J}\, dz^I \wedge d\bar{z}^J$$

then

$$\partial G = \sum_{I \in H_r, J \in H_s} \partial G_{I,J} \wedge dz^I \wedge d\bar{z}^J$$

and

$$\bar{\partial} G = \sum_{I \in H_r, J \in H_s} \bar{\partial} G_{I,J} \wedge dz^I \wedge d\bar{z}^J.$$

In the same way as in the classical case we have $\bar{\partial}^2 G = \bar{\partial}(\bar{\partial} G) = 0$. If U and V are two open subsets of $\mathbb{C}^n$ with $U \subset V$ then we have a natural restriction map $J^V_U\colon \mathscr{G}^*(V) \to \mathscr{G}^*(U)$. This restriction map extends trivially to the case of (p, q) forms. If $G \in \mathscr{G}^*_{(r,s)}(\Omega)$ we define as usual the support of G by means of the support of its coefficients $G_{I,J} \in \mathscr{G}^*(\Omega)$.

9. THE EQUATION $\bar{\partial} S = G$, $G \in \mathscr{G}^*_{(0,1)}(\mathbb{C}^n)$ WITH COMPACT SUPPORT

The followings result is the n-dimensional extension of Lemma 3 of Section 6. In the sequel we set $g(\psi, z_1, \ldots, z_n)$ instead of $g(\psi, (z_1, \ldots, z_n))$ when $g \in \mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}^n]$.

LEMMA 1. *Let be given $G \in \mathscr{G}^*(\mathbb{C}^n)$ with compact support K, L a compact subset of $\mathbb{C}^n$ containing K in its interior and let $g \in \mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}^n]$ be the representative of G associated to L obtained in Lemma 2 of Section 6. Then the following formula defines a function $g^*\colon \mathscr{A}_1 \times \mathbb{C}^n \to \mathbb{C}$*

$$g^*(\psi, z) = \frac{1}{2\pi i} \int g(\psi, \zeta, z_2, \ldots, z_n) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z_1} \tag{19}$$

($\psi \in \mathscr{A}_1$, $z = (z_1, z_2, \ldots, z_n)$) and g^* has the following properties

(a) $g^* \in \mathscr{E}[\mathscr{A}_1 \times \mathbb{C}^n]$,

(b) $\partial g^*/\partial \bar{z}_1 = g$ *in* $\mathscr{A}_1 \times \mathbb{C}^n$,

(c) $(\partial g^*/\partial \bar{z}_k)(\psi, z) = (1/2\pi i) \int (\partial g/\partial z_k)(\psi, z_1 + \zeta, z_2, ..., z_n)(d\zeta \wedge d\bar{\zeta}/\zeta)$ if $k > 1$

(d) $g^* \in \mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}^n]$.

Proof. Cases (a) and (b) follow immediately from the proof of Lemma 3 of Section 6, considering $(z_2, ..., z_n)$ as a parameter. The differentiability in the real variables corresponding to $z_2, ..., z_n$, Case (c) and the fact that g^* is moderate follow from the formula

$$g^*(\varphi_\varepsilon, z) = \frac{1}{2\pi i} \int g(\varphi_\varepsilon, z_1 + \zeta, z_2, ..., z_n) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta}$$

since g is moderate, has compact support independent of $\varepsilon > 0$ and that $1/|\zeta|$ is locally integrable in $\mathbb{R}^{2n}$ $(n \geqslant 1)$. ∎

Now we generalize Theorem 2 to the n-variable case:

THEOREM 4. *Let be given $G \in \mathscr{G}^*_{(0,1)}(\mathbb{C}^n)$ with compact support and such that $\bar{\partial} G = 0$. Then there is $S \in \mathscr{G}^*(\mathbb{C}^n)$ such that $\bar{\partial} S = G$.*

Proof. $G = \sum_{j=1}^n G_j \, d\bar{z}_j$ with $G_j \in \mathscr{G}^*(\mathbb{C}^n)$. We consider a compact subset L of $\mathbb{C}^n$ whose interior contains the support of each G_j. From the lemma above, if g_1 is a representative of G_1, we define $g_1^* \in \mathscr{E}_M[\mathscr{A}_1 \times \mathbb{C}^n]$ by

$$g_1^*(\psi, z) = \frac{1}{2\pi i} \int g_1(\psi; \zeta, z_2, ..., z_n) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z_1}$$

which has the properties listed there. We define $S \in \mathscr{G}^*(\mathbb{C}^n)$ as the class of g_1^* and in order to obtain $\bar{\partial} S = G$ it suffices to prove that

$$\frac{\partial g_1^*}{\partial \bar{z}_k} - g_k \in \mathscr{N}[\mathscr{A}_1 \times \mathbb{C}^n] \tag{20}$$

(Definition 4) for all $k = 1, ..., n$ if g_k is a representative of G_k. We choose g_k such that for all $\psi \in \mathscr{A}_1$ the function $z \to g_k(\psi, z)$ has support in L° (see Lemma 2 of Section 6). Since $\bar{\partial} G = 0$ it follows that for all $k, j = 1, 2, ..., n$

$$\eta_{j,k} = \frac{\partial g_j}{\partial \bar{z}_k} - \frac{\partial g_k}{\partial \bar{z}_j} \in \mathscr{N}[\mathscr{A}_1 \times \mathbb{C}^n].$$

In particular for $j = 1$ we have

$$\frac{\partial g_1}{\partial \bar{z}_k} = \frac{\partial g_k}{\partial \bar{z}_1} + \eta_{1,k}. \tag{21}$$

Therefore from formula (c) of Lemma 1 we have

$$\frac{\partial g_1^*}{\partial \bar z_k}(\psi, z) = \frac{1}{2\pi i}\int \frac{\partial g_k}{\partial \bar z_1}(\psi, \zeta, z_2, ..., z_n)\frac{d\zeta \wedge d\bar\zeta}{\zeta - z_1} + \frac{1}{2\pi i}\int \eta_{1,k}(\psi, \zeta, z_2, ..., z_n)\frac{d\zeta \wedge d\bar\zeta}{\zeta - z_1}. \tag{22}$$

Now for all $\psi \in \mathscr{A}_1$ the function $z \to g_k(\psi, z)$ has its support contained in $\mathring{L}$. Let D be a disk in $\mathbb{C}$ of center 0 and of radius large enough so that $g_k(\psi, z)$ is null if $z \in \partial D$. From the Cauchy formula for C^∞ functions (Hörmander [6, Theorem 1.2.1])

$$2\pi i g_k(\psi, z) = \int_{\partial D} \frac{g_k(\psi, \zeta, z_2, ..., z_n)}{\zeta - z_1}\, d\zeta + \int_D \frac{\partial g_k}{\partial \bar z_1}(\psi, \zeta, z_2, ..., z_n)\frac{d\zeta \wedge d\bar\zeta}{\zeta - z_1}.$$

Since the first integral is null (from the choice of D), (22) becomes

$$\frac{\partial g_1^*}{\partial \bar z_k}(\psi, z) - g_k(\psi, z) = \frac{1}{2\pi i}\int \eta_{1,k}(\psi, \zeta, z_2, ..., z_n)\frac{d\zeta \wedge d\bar\zeta}{\zeta - z_1} = -\frac{1}{2\pi i}\int \eta_{1,k}(\psi, z_1 - \zeta, z_2, ..., z_n)\frac{d\zeta \wedge d\bar\zeta}{\zeta}. \tag{23}$$

Now (20) follows from (21) and (23) since $\eta_{1,k}$ has compact support independent of $\psi \in \mathscr{A}_1$ and since $1/|\zeta|$ is locally integrable in $\mathbb{R}^{2n}$. ∎

10. The Dolbeault–Grothendieck Lemma

Lemma 1. *Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$ and $\Omega' = \Omega'_1 \times \cdots \Omega'_n$ be two open polydiscs of $\mathbb{C}^n$ with the same polycenter such that $\bar\Omega'$ is compact and*

$$\bar\Omega' \subset \Omega.$$

Let $k \in \mathbb{N}$ be such that $0 \leqslant k \leqslant n$. Then for each $g \in \mathscr{G}^*(\Omega)$ such that

$$\frac{\partial g}{\partial \bar z_j} = 0 \qquad \text{in } \Omega \text{ if } j > k \tag{24}$$

(this condition is void if $k = n$) there exists $G \in \mathscr{G}^*(\Omega)$ such that

$$\frac{\partial G}{\partial \bar z_k} = g \qquad \text{in } \Omega' \qquad \text{and} \qquad \frac{\partial G}{\partial \bar z_j} = 0 \qquad \text{in } \Omega \text{ if } j > k. \tag{25}$$

Proof. Let $\alpha \in \mathscr{D}(\Omega_k) \subset \mathscr{D}(\mathbb{R}^2)$ be such that

$$0 \leqslant \alpha \leqslant 1 \qquad \text{and} \qquad \alpha \equiv 1 \qquad \text{in } \bar{\Omega}'_k.$$

Let $\tilde{g} \in \mathscr{E}_M[\mathscr{A}_1 \times \Omega]$ be a representative of g. For each $\psi \in \mathscr{A}_1$ and $z = (z_1, ..., z_n) \in \Omega$ we define:

$$\tilde{G}(\psi, z) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{\alpha(\zeta)\, \tilde{g}(\psi, z_1, ..., z_{k-1}, \zeta, z_{k+1}, ..., z_n)}{\zeta - z_k}\, d\zeta \wedge d\bar{\zeta}$$
$$= -\frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{\alpha(z_k - \zeta)\, \tilde{g}(\psi, z_1, ..., z_{k-1}, z_k - \zeta, z_{k+1}, ..., z_n)}{\zeta}\, d\zeta \wedge d\bar{\zeta}. \tag{26}$$

As usual (26) shows that $\tilde{G} \in \mathscr{E}[\mathscr{A}_1 \times \Omega]$. As already done several times on similar expressions (the presence of the function α does not cause any trouble using Leibnitz's formula for derivation of a product) one proves that $\tilde{G} \in \mathscr{E}_M[\mathscr{A}_1 \times \Omega]$. From (26) and the Lemma 1 of Section 6 we have

$$\frac{\partial}{\partial \bar{z}_k} \tilde{G}(\psi, z_1, ..., z_{k-1}, \zeta, z_{k+1}, ..., z_n)$$
$$\alpha(\zeta)\, \tilde{g}(\psi, z_1, ..., z_{k-1}, \zeta, z_k, ..., z_n)$$

if $\zeta \in \mathbb{R}^2$, $\psi \in \mathscr{A}_1$ and $z_j \in \Omega_j$ if $j \neq k$. Since $\alpha \equiv 1$ in Ω'_k it follows that

$$\frac{\partial \tilde{G}}{\partial \bar{z}_k}(\psi, z) = \tilde{g}(\psi, z) \tag{27}$$

if $z \in \Omega'$ and $\psi \in \mathscr{A}_1$. Since $\partial \tilde{g}/\partial \bar{z}_j \in \mathscr{N}[\mathscr{A}_1 \times \Omega]$ if $j > k$ (from (24)) it follows as usual from (26) that

$$\frac{\partial \tilde{G}}{\partial \bar{z}_j} \in \mathscr{N}[\mathscr{A}_1 \times \Omega] \tag{28}$$

if $j > k$. Defining $G \in \mathscr{G}^*(\Omega)$ as the class of $\tilde{G}$, (25) follows immediately from (27) and (28). ∎

THEOREM 5. *Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$ be an open polydisc in $\mathbb{C}^n$ and $f \in \mathscr{G}^*_{(p, q+1)}(\Omega)(p, q \geqslant 0)$ such that $\bar{\partial} f = 0$. Let W be an open subset of Ω whose closure $\bar{W}$ is contained and compact in Ω. Then there exists $u \in \mathscr{G}^*_{(p,q)}(\Omega)$ such that $\bar{\partial} u = f$ in W.*

Proof. As in the classical case (Hörmander [6, Theorem 2.3.3]) this is proved by induction on a number $k \in \mathbb{N}$ such that f is independent on $d\bar{z}_{k+1}, ..., d\bar{z}_n$. It suffices to follow the classical proof and use the above lemma. ∎

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